Universality of probability distributions among two-dimensional turbulent flows

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We study statistical properties of two-dimensional turbulent flows. Three systems are considered: the Navier-Stokes equation, surface quasigeostrophic flow, and a model equation for thermal convection in the Earth's mantle. Direct numerical simulations are used to determine one-point fluctuation properties. Comparative study shows universality of probability density functions (PDFs) across different types of flow. For instance, the PDFs for derivatives of the advected quantity are the same for the three flows, once normalized by the average size of fluctuations. The single-point statistics is surprisingly robust with respect to the nature of the nonlinearity.

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The central idea of classical turbulence theory is that certain statistical properties in turbulent flow are independent of the details of the flow, like its boundaries, dissipation mechanism, and the kind of forcing, as long as the Reynolds number is sufficiently high [1]. In this sense turbulent flow would be universal. In this article we shall investigate independence not of boundary conditions, dissipation, or forcing, but look for universality across *equations*. This idea has long been demonstrated to hold for several classes of partial differential equations. Certain partial differential equations, sharing common symmetries, exhibit identical fluctuation properties, once normalized by the standard deviations. (This is well known from ''quantum chaos'' and ''wave chaos'' [2]). In the present article it is demonstrated that three different equations describing fluid flow and subjected to the same external conditions exhibit the same statistics for their fluctuations, once normalized by the average size of fluctuations.

The three flows are described by advection-diffusion equations

$$\frac{\partial\theta}{\partial t}+\vec{v}\cdot\nabla\theta=D\nabla^2\theta+f. \tag{1a}$$

The scalar quantity advected is $\theta(\vec{r},t)$. The vector $\vec{r}$ describes the spatial location. The forcing $f(\vec{r},t)$ supplies the energy dissipated via a dissipation constant D. The velocity $\vec{v}(\vec{r},t)$ is a function of θ, best written in Fourier space,

$$\hat{\vec{v}}(\vec{k},t)=i\frac{\vec{k}\times\hat{\theta}(\vec{k},t)}{|\vec{k}|^{\alpha}}. \tag{1b}$$

The two-dimensional cross product $\vec{k}\times\hat{\theta}$ is to be understood as a vector of length $|\vec{k}\hat{\theta}|$ and direction perpendicular to $\vec{k}$.

Different values of α correspond to different flows [3]. The two-dimensional Navier-Stokes equation is $\alpha=2$ and θ corresponds to the vorticity $\nabla\times\vec{v}$. The surface quasigeostrophic equation, $\alpha=1$, is a special case of the important quasigeostrophic equation that describes flow of a shallow layer on a rotating sphere, as relevant for planetary atmospheres and oceans [4]. In this case, θ is physically interpreted as temperature, which drives the flow through its buoyancy effect. The third equation considered is $\alpha=3$, which also appears in geophysical context as a limiting case of a shallow flow on a rotating sphere with uniform internal heating [5]. Also here, θ is an (active) temperature.

Other values of α, integer or not, could be considered, but this is not done here. There are numerous studies for $\alpha=1$ (e.g., Refs. [4,6–8]) and, of course, for the Navier-Stokes equation, while other values of α have received less attention [9,10]. Analytical comparisons between the $\alpha=1$ and $\alpha=2$ case are provided in Refs. [7,11,3].

Multiplying Eq. (1a) with θ and averaging over space with periodic boundary conditions yields

$$\frac{1}{2}\frac{\partial}{\partial t}\langle\theta^2\rangle=-D\langle|\nabla\theta|^2\rangle+\langle f\theta\rangle. \tag{2}$$

Consequently, the left-hand side of Eq. (1a) conserves $\langle\theta^2\rangle$ for all α. For $\alpha=2$ also $\langle\vec{v}^2\rangle$ is a conserved quantity, while $\langle\vec{v}^2\rangle$ is identical to $\langle\theta^2\rangle$ for $\alpha=1$, and $\langle\vec{v}^2\rangle$ is not conserved for $\alpha=3$. Equation (1) is invariant under reflection, $\vec{r}\to-\vec{r}$, as well as the set of simultaneous transformations $\vec{r}\to\vec{r}\lambda$, $t\to t\lambda^2$, $\theta\to\theta/\lambda^{\alpha}$, and $f\to f/\lambda^{\alpha+2}$ (this is essentially the Reynolds number invariance). The family of flows with different α has been named α turbulence [4], although this term appears in the literature also for other kinds of flow. We shall consider isotropic, homogeneous, and statistically stationary turbulence with white-in-time forcing.

The flow is simulated in a doubly periodic square box. Fourier modes are labeled $|\vec{k}|=1$ if their wavelength equals the box size. Forcing acts on large scales, $4\leqslant|\vec{k}|<6$, with constant amplitude but random phases renewed at each time step. The time step is held constant. Two-dimensional turbulent flows produce vortices that merge and grow ever larger. This vorticity must be removed in order to reach an equilibrium state. This is done by adding a large-scale dissipation $-\gamma\theta$ to the right-hand side of Eq. (1a), restricted to $0<|\vec{k}|\leqslant3$. The parameter $1/\gamma$ is much larger than other time scales of motion, so that the lowest modes decay gradually. The simulations are carried out with a pseudospectral method over long time periods using fourth-order Runge-Kutta integration of the Fourier modes. A mild spectral filter is used, without complete dealiasing, since it is not clear whether complete dealiasing improves or worsens the quality of simulations. Further details about the numerics are given elsewhere [10].

The aforementioned invariance naturally defines a Reynolds number for flow of any α as $\mathrm{Re}=UL/D$, where U and L are a velocity and length scale, respectively. We choose $U=\sqrt{\langle \vec{v}^2\rangle}$ and $L=1$ for a large-scale Reynolds number. With this definition the maximum Reynolds numbers achieved are on the order of several thousands on a 1024×1024 grid for each of the three flows.

In this article only one-point probability density functions (PDFs) are studied. First, the Navier-Stokes equation ($\alpha=2$) is treated, which is important by itself and also exemplifies the variations and dependencies in the PDFs within one equation. Second, equations with different values of α are compared with each other, which is the central concern of this article.

The PDFs are obtained from spatial averaging of the flow field and additional time averaging over 8–24 such snapshots. Whenever PDFs are compared with each other in the subsequent figures they are scaled by their average fluctuation, defined as

$$\sigma=\int dx|x|P(x). \tag{3}$$

The integral is over all x. Instead of the first absolute moment (3) the standard deviation could be used as well. All PDFs of the Navier-Stokes equation presented here agree with the ones reported from recent simulations by Takahashi and Gotoh [12] at higher Reynolds numbers.

Figure 1 shows PDFs for different Reynolds numbers. In Fig. 1(a) we see similar but not at all identical shapes for the PDFs, a behavior representative for the PDFs of other quantities as well.

According to Fig. 2(a) velocity components are distributed Gaussian. The PDFs for v_x and v_y are almost identical, as must be true for isotropic turbulence. If the two velocity components are statistically independent of each other, then the PDF of the absolute value of $\vec{v}$ should be a two-dimensional Maxwell distribution

$$P(x)=\frac{x}{s^2}\exp\left(-\frac{x^2}{2s^2}\right). \tag{4}$$

The parameter s is thereby the standard deviation of the Gaussian distribution for v_x. The Maxwell distribution plotted as dotted curve in Fig. 2(b) hence contains no free parameter. It is a good first-order approximation.

As a matter of space not all PDFs can be presented here. The scalar (vorticity) is Gaussian in the center. The longitudinal velocity derivatives are also Gaussian. This is particularly striking, since velocity derivatives of decaying turbulence are not Gaussian [13,14]. Their core behaves much more like a Cauchy distribution

$$P(x)=\frac{1}{\pi}\frac{c}{c^2+x^2},$$

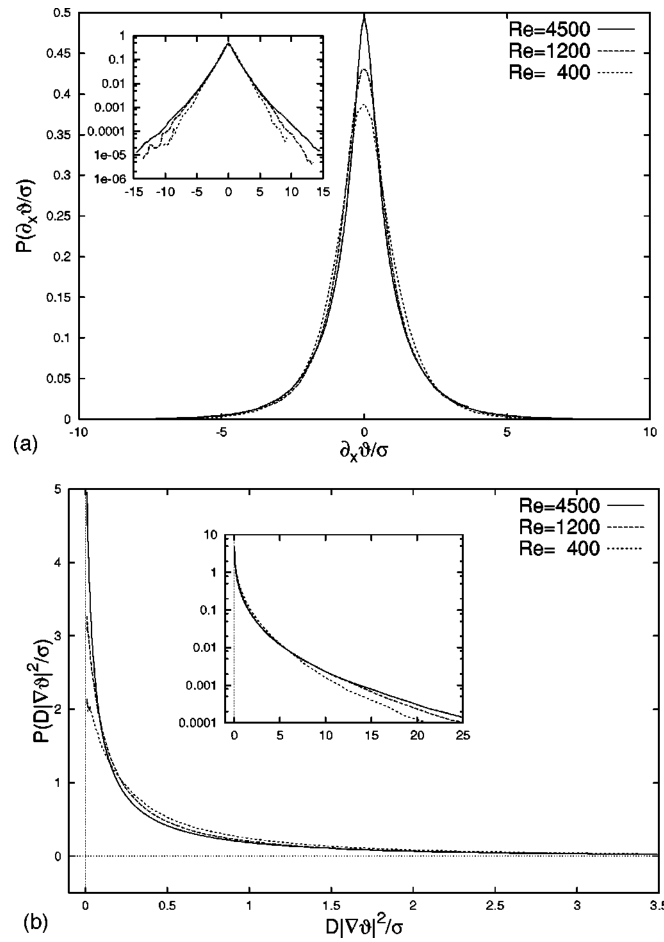

FIG. 1. Probability density functions for the Navier-Stokes equation at different Reynolds numbers (a) vorticity derivative $\partial_x\theta$ and (b) vorticity dissipation $D|\nabla\theta|^2$. Many small fluctuations account for most of the dissipation. The insets show the same data on a logarithmic scale.

which has an inflection point even on a logarithmic plot. A Cauchy distribution follows theoretically from a ‘‘dilute gas’’ of point vortices of equal strength that move randomly, see Ref. [13]. For forced two-dimensional turbulence $\partial_x\theta$ and $\nabla^2\theta$, for example, possess an inflection point on a logarithmic scale. (Point vortex models cannot make any sensible predictions on the scalar derivatives.)

PDFs in α turbulence have been previously reported from simulations at lower resolution in [15] (Fig. 7), where a ‘‘remarkable similarity’’ has been pointed out for the PDFs of one of the variables (the scalar θ).

Figure 3 shows PDFs for different types of flow. In each figure the PDFs of all three flows are shown simultaneously, and the different figures show scalar derivative $\partial_x\theta$, scalar dissipation $D|\nabla\theta|^2$, and velocity component v_x. Apparently the PDFs for the different flows are the same. The agreement is for small as well as large fluctuations up to several standard deviations. The deviations in the far tails could be fundamental or they could lie within measurement errors, since the very largest fluctuations are inevitably undersampled.

Not all PDFs overlap as accurately as the derivative of θ. The deviations in Fig. 3(c) are somewhat larger. Other PDFs show even larger deviations, but in none of the investigated variables is there any drastic difference. The PDFs for $\partial_x\theta$,

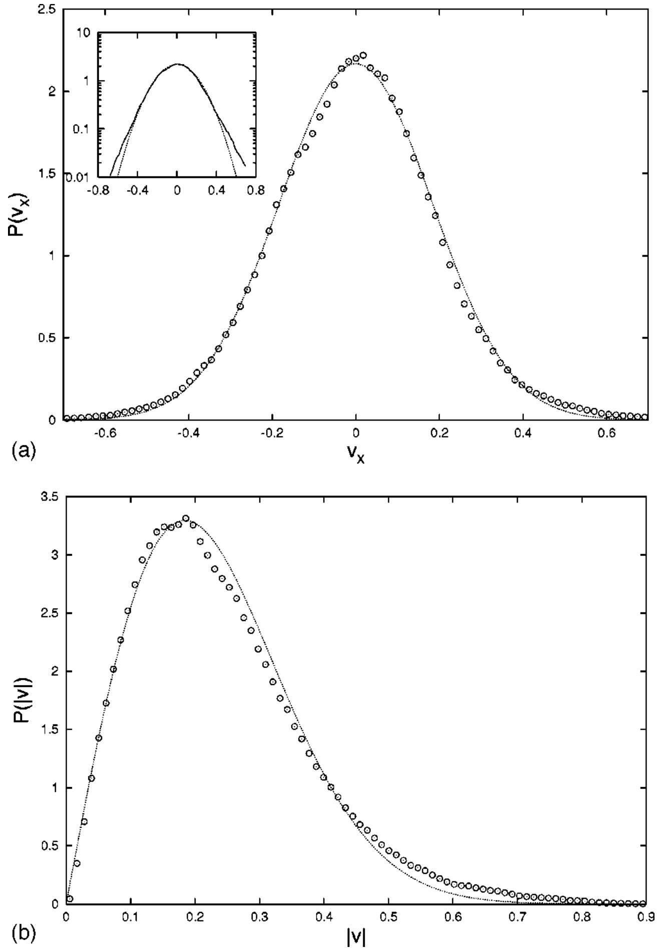

FIG. 2. Probability density functions for velocities of the Navier-Stokes at Re = 4500. (a) Velocity component v_x and (b) absolute value of velocity $|\vec{v}|$. The dotted lines are theoretical fits (Gaussian and Maxwellian).

$|\nabla\theta|^2$, v_x (and $\partial_y\theta$, v_y) closely agree over several standard deviations of the respective variables. Slightly worse, but still amazing agreement is seen for θ, $\nabla^2\theta$, $|v|$ and $\nabla\times v$. It is conceivable that these deviations lie within measurement errors. Some PDFs require longer averaging for convergence than others. Whether these deviations are fundamental or due to insufficient or fundamental statistics cannot be decided with the data at hand and it remains therefore unclear whether the universality extends to all local variables or not.

The PDFs of Fig. 3 are related to each other by a simple rescaling, except perhaps for very large fluctuations. This establishes an invariance of PDFs with respect to the relation between velocity and scalar and hence with respect to the nature of the nonlinearity.

The Reynolds numbers in the simulations for $\alpha=1,2,3$ are Re = 3900, 4500, 4200, respectively. At sufficiently high Reynolds number one does not expect any remaining Reynolds number dependence. Convergent PDFs, normalized on the standard deviation, have indeed been reported in Ref. [12] at Reynolds numbers around 10 000. There is a weak dependence on the Reynolds number for the PDFs used in this study. In fact, Fig. 1 demonstrates this. Consequently, the flows should be compared at equal Reynolds number. In the simulations the Reynolds number is only determined in retrospective and hence they are not precisely equal. However, since the Reynolds number dependence is slow an approximate agreement suffices. That is to say, the above differences in the Reynolds numbers are not significant. Comparisons at a lower Reynolds number (around 1300) yield the same universalities.

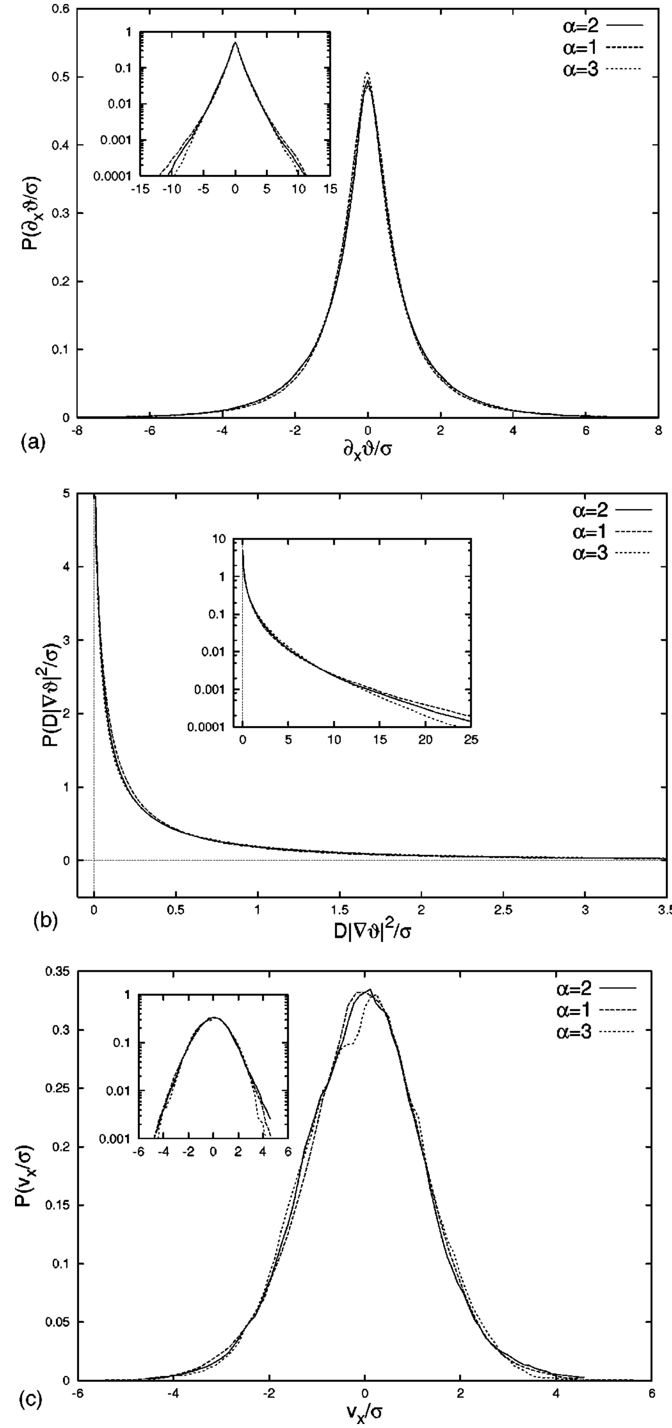

FIG. 3. The main result. Probability density functions for different α. The flows have the same forcing and similar Reynolds number. (a) The gradient $\partial_x\theta$, (b) the scalar dissipation $D|\nabla\theta|^2$, and (c) the velocity component v_x. The PDFs are divided by their respective average fluctuation σ. The shape of the PDFs is independent of the type of flow.

The flows have been compared for one set of conditions. In particular, the same boundary conditions and forcing are used for all three flows. One can imagine changing the time correlation of the forcing, the order of the dissipation, or the boundary conditions among other possibilities. It is *not* claimed here that the probability functions will remain the same. For instance, a moving boundary will certainly alter at least some of the PDFs. However, based on the evidence presented here one would expect the altered PDFs to be again identical among the different flows. The finding here is one of robustness with respect to the dynamics of the flow, but not necessarily with respect to other alterations.

The three equations describe physically different flows. Also their mathematical properties differ in important ways. For example, the stretching of contour lines, conservation of kinetic energy, and the locality of a presumable enstrophy cascade [9,10] change with α. In spite of these immense differences, the fluctuation properties vary only in absolute size. Of course, there are also commonalities among the different flows, namely, the number of quadratically conserved quantities, incompressibility, reflection symmetry, and Reynolds number invariance. It remains an open question how far the universality generalizes to other kinds of advection-diffusion equations with a different type of velocity-scalar relation than investigated here.

To summarize, several probability density functions of three different flows have identical shapes. These flows are described by advection equations with an incompressible velocity field, reflection symmetry, and Reynolds number invariance. They are compared at identical external conditions (same boundary conditions and forcing) and have the same dissipation mechanism. For the family of flows described, the particular nature of the nonlinearity does not alter the probability of fluctuations, besides their average amplitude. This demonstrates that fluctuations in fully developed two-dimensional turbulence arise from a simple and general mechanism, perhaps of purely statistical nature, since specific dynamics does not matter. No corresponding claim is made about other important statistical properties, such as spatial correlations or spectra [10], which are not single-point properties.

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